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and mixed Hodge structures**

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# Cup products and mixed Hodge structures

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## Abstract

Let  $X$  be a complete complex algebraic variety of dimension  $n$  and let  $D$  be a divisor with strict normal crossings on  $X$ . In this paper we show that the cup product maps

$$H^i(X \setminus D) \otimes H^j(X, D) \rightarrow H^{i+j}(X, D)$$

and

$$H_D^i(X) \otimes H^j(D) \rightarrow H_D^{i+j}(X)$$

are morphisms of mixed Hodge structures.

## 1 Introduction

Let  $X$  and  $Y$  be topological spaces,  $A$  a commutative ring,  $\mathcal{K}^\bullet$  a complex of sheaves of  $A$ -modules on  $X$  and  $\mathcal{L}^\bullet$  a complex of sheaves of  $A$ -modules on  $Y$ . Then one has the complex of sheaves of  $A$ -modules  $\mathcal{K}^\bullet \boxtimes \mathcal{L}^\bullet$  on  $X \times Y$  and canonical homomorphisms

$$\mathbb{H}^p(X, \mathcal{K}^\bullet) \otimes_A \mathbb{H}^q(Y, \mathcal{L}^\bullet) \rightarrow \mathbb{H}^{p+q}(X \times Y, \mathcal{K}^\bullet \boxtimes \mathcal{L}^\bullet) .$$

In the case  $X = Y$ , suppose that moreover we are given a homomorphism of complexes of sheaves of  $A$ -modules

$$\mathcal{K}^\bullet \otimes \mathcal{L}^\bullet \rightarrow \mathcal{M}^\bullet$$

then, using that the restriction of  $\mathcal{K}^\bullet \boxtimes \mathcal{L}^\bullet$  to the diagonal of  $X \times X$  is equal to  $\mathcal{K}^\bullet \otimes \mathcal{L}^\bullet$ , one obtains homomorphisms

$$\mathbb{H}^p(X, \mathcal{K}^\bullet) \otimes_A \mathbb{H}^q(X, \mathcal{L}^\bullet) \rightarrow \mathbb{H}^{p+q}(X, \mathcal{M}^\bullet) .$$

As an example, we let  $X$  be a smooth complete complex algebraic variety, and consider the map

$$\mu : \Omega_X^\bullet \otimes_{\mathbb{C}} \Omega_X^\bullet \rightarrow \Omega_X^\bullet$$

given by cup product of forms. It induces the cup product on cohomology with coefficients in  $\mathbb{C}$  and we can deduce Poincaré duality from Serre duality in the following way. We let  $n = \dim X$  and focus on the cup product

$$\mathbb{H}^k(X, \Omega_X^\bullet) \otimes_{\mathbb{C}} \mathbb{H}^{2n-k}(X, \Omega_X^\bullet) \rightarrow \mathbb{H}^{2n}(X, \Omega_X^\bullet) \simeq \mathbb{C} .$$

Note that the morphism  $\mu$  maps  $\sigma^{\geq p}\Omega_X^\bullet \otimes \sigma^{\geq q}\Omega_X^\bullet$  to  $\sigma^{\geq p+q}\Omega_X^\bullet$  so the induced map

$$F^p\mathbb{H}^k(X, \Omega_X^\bullet) \otimes_{\mathbb{C}} F^q\mathbb{H}^{2n-k}(X, \Omega_X^\bullet) \rightarrow \mathbb{H}^{2n}(X, \Omega_X^\bullet)$$

is the zero map as soon as  $p + q > n$ . For  $p + q = n$  the pairing

$$Gr_F^p\mathbb{H}^k(X, \Omega_X^\bullet) \otimes_{\mathbb{C}} Gr_F^q\mathbb{H}^{2n-k}(X, \Omega_X^\bullet) \rightarrow \mathbb{H}^{2n}(X, \Omega_X^\bullet)$$

is nonsingular by Serre duality. This implies that the pairing

$$\mathbb{H}^k(X, \Omega_X^\bullet) \otimes_{\mathbb{C}} \mathbb{H}^{2n-k}(X, \Omega_X^\bullet) \rightarrow \mathbb{H}^{2n}(X, \Omega_X^\bullet) \simeq \mathbb{C}.$$

is also nonsingular.

Let  $X$  be a complete complex algebraic variety of dimension  $n$  and let  $D$  be a divisor with strict normal crossings on  $X$ . In this paper we show that the cup product maps

$$H^i(X \setminus D) \otimes H^j(X, D) \rightarrow H^{i+j}(X, D)$$

and

$$H_D^i(X) \otimes H^j(D) \rightarrow H_D^{i+j}(X)$$

(the "extraordinary cup product, cf. [5, p. 127] are morphisms of mixed Hodge structures. A consequence of these facts is Fujiki's theorem [4]: if  $D$  is a strict normal crossing divisor on the smooth complete variety  $X$  and  $n = \dim(X)$ , then the cup products

$$H^i(X \setminus D) \otimes H^{2n-i}(X, D) \rightarrow H^{2n}(X, D) \simeq \mathbb{Q}(-n)$$

and

$$H_D^i(X) \otimes H^{2n-i}(D) \rightarrow H_D^{2n}(X) \simeq \mathbb{Q}(-n)$$

induce dualities of mixed Hodge structures, and the exact sequences of mixed Hodge structures

$$\cdots \rightarrow H^i(X, D) \rightarrow H^i(X) \rightarrow H^i(D) \rightarrow H^{i+1}(X, D) \rightarrow \cdots$$

$$\cdots \leftarrow H^{2n-i}(X \setminus D) \leftarrow H^{2n-i}(X) \leftarrow H_D^{2n-i}(X) \leftarrow H^{2n-i-1}(X \setminus D) \leftarrow \cdots$$

are dual to each other (with respect to  $\mathbb{Q}(-n)$ ).

## 2 Some cohomological mixed Hodge complexes

A *cohomological mixed Hodge complex* on a variety  $X$  consists of data

- $(K_{\mathbb{Q}}^\bullet, W)$ , a complex of sheaves of  $\mathbb{Q}$ -vector spaces on  $X$  with an increasing filtration  $W$ ;
- $(K_{\mathbb{C}}^\bullet, W, F)$ , a complex of sheaves of  $\mathbb{C}$ -vector spaces on  $X$  with an increasing filtration  $W$  and a decreasing filtration  $F$ ;

- an isomorphism  $\alpha : (K_{\mathbb{Q}}^{\bullet}, W) \otimes \mathbb{C} \simeq (K_{\mathbb{C}}^{\bullet}, W)$  in the filtered derived category  $DF(X, \mathbb{C})$  of sheaves of  $\mathbb{C}$ -vector spaces on  $X$  with an increasing filtration

which satisfy certain axioms, see [2, Sect. 8]. Such an object gives rise to mixed  $\mathbb{Q}$ -Hodge structures on the hypercohomology groups  $\mathbb{H}^k(X, K_{\mathbb{C}}^{\bullet})$ : the  $\mathbb{Q}$ -structure is induced by the map defined by  $\alpha$  on hypercohomology, and the weight and Hodge filtrations are induced by  $W$  and  $F$  respectively.

Let  $X$  be a smooth complete variety of dimension  $n$  and let  $D = \bigcup_{\alpha \in A} D_{\alpha}$  be a divisor with strict normal crossings on  $X$ . Let  $U = X \setminus D$ . Then one has a mixed Hodge structure on  $H^k(U)$  defined by the "standard" cohomological mixed Hodge complex  $K^{\bullet}(X \log D)$  whose  $\mathbb{C}$ -component satisfies

$$K^{\bullet}(X \log D)_{\mathbb{C}} = \Omega_X^{\bullet}(\log D)$$

with  $W$  and  $F$  as in [1, Sect. 3.1].

Similarly, one has a mixed Hodge structure on  $H^k(D)$  defined by a standard cohomological mixed Hodge complex  $K^{\bullet}(D)$  which is described as follows.

Let  $D_{\bullet}$  be the semisimplicial variety with  $D_m$  the union of  $(m+1)$ -fold intersections of components of  $D$ . We have an augmentation  $\pi_{\bullet} : D_{\bullet} \rightarrow D$  which is of cohomological descent: for any complex of sheaves of abelian groups  $F^{\bullet}$  on  $D$  one has a quasi-isomorphism  $F^{\bullet} \rightarrow \pi_{*}\pi^{*}F^{\bullet}$ . (Here  $\pi_{*}\pi^{*}F^{\bullet}$  is the single complex associated to the double complex  $\bigoplus_{p,q} (\pi_p)_{*}(\pi_p)^{*}F^q$ ). Then

$$K^{\bullet}(D)_{\mathbb{Q}} = \pi_{*}\pi^{*}\mathbb{Q}_D = \bigoplus_p (\pi_p)_{*}\mathbb{Q}_{D_p}$$

with

$$W_k K^{\bullet}(D)_{\mathbb{Q}} = \bigoplus_{p \geq -k} (\pi_p)_{*}\mathbb{Q}_{D_p}$$

and

$$K^{\bullet}(D)_{\mathbb{C}} = \bigoplus_p (\pi_p)_{*}\Omega_{D_p}^{\bullet}$$

with the filtration  $W$  as in the rational case and the filtration  $F$  given by

$$F^q K^{\bullet}(D)_{\mathbb{C}} = \bigoplus_p (\pi_p)_{*} F^q \Omega_{D_p}^{\bullet} .$$

Define  $\tilde{\Omega}_D^{\bullet} = \Omega_D^{\bullet} \bmod \text{torsion}$ . Then

$$\tilde{\Omega}_D^{\bullet} \simeq \Omega_X^{\bullet} / \Omega_X^{\bullet}(\log D)(-D)$$

with the filtration  $F$  induced from  $\Omega_X^{\bullet}$ , and one has a filtered quasi-isomorphism

$$(\tilde{\Omega}_D^{\bullet}, F) \simeq (K^{\bullet}(D)_{\mathbb{C}}, F) .$$

However, there is no weight filtration on  $\tilde{\Omega}_D^{\bullet}$ .

One has a natural restriction map of cohomological mixed Hodge complexes  $i^* : K^\bullet(X) \rightarrow K^\bullet(D)$ , and the mixed Hodge structure on  $H^*(X, D)$  is obtained from the cohomological mixed Hodge complex

$$K^\bullet(X, D) := \text{Cone}(i^*)[-1]$$

(here we take the so-called *mixed cone*, cf. [3, Sect. 3.3], and  $[-1]$  denotes a shift of index in the complex). Observe that

$$(K^\bullet(X, D)_\mathbb{C}, F) \simeq \ker((\Omega_X^\bullet, F) \rightarrow (\tilde{\Omega}_D^\bullet, F)) \simeq (\Omega_X^\bullet(\log D)(-D), F)$$

This observation suffices to show that the cup product mapping

$$H^r(X \setminus D) \otimes H^s(X, D) \rightarrow H^{r+s}(X, D)$$

is compatible with the Hodge filtrations. Indeed, cup product induces a morphism of complexes

$$\Omega_X^\bullet(\log D) \otimes_\mathbb{C} \Omega_X^\bullet(\log D)(-D) \rightarrow \Omega_X^\bullet(\log D)(-D)$$

which is compatible with the induced Hodge filtrations. Moreover observe that  $\Omega_X^n(\log D)(-D) \simeq \Omega_X^n$  so  $\mathbb{H}^{2n}(X, \Omega_X^\bullet(\log D)(-D)) \simeq H^n(X, \Omega_X^n) \simeq \mathbb{C}$  if  $X$  is connected. Now the argument runs just like in the smooth projective case: Serre duality gives a non-degenerate pairing between

$$Gr_F^p H^k(X \setminus D) \simeq H^{k-p}(X, \Omega_X^p(\log D))$$

and

$$Gr_F^{n-p} H^{2n-k}(X, D) \simeq H^{n-k+p}(X, \Omega_X^{n-p}(\log D)(-D)) .$$

However, there is no weight filtration on  $\Omega_X^\bullet(\log D)(-D)$ , so we cannot use the complex  $\Omega_X^\bullet(\log D)(-D)$  to prove compatibility of the cup product with the weight filtration.

### 3 A weak equivalence

The cup product map

$$H^i(X \setminus D, \mathbb{C}) \otimes_\mathbb{C} H^j(X, D; \mathbb{C}) \rightarrow H^{i+j}(X, D; \mathbb{C})$$

by the results of Sect. 2 is reformulated as a map

$$\mathbb{H}^i(X, \Omega_X^\bullet(\log D)) \otimes_\mathbb{C} \mathbb{H}^j(X, \Omega_{X,D}^\bullet) \rightarrow \mathbb{H}^{i+j}(X, \Omega_{X,D}^\bullet) .$$

However, we do not dispose of a natural cup product mapping on the level of complexes

$$\Omega_X^\bullet(\log D) \otimes_{\mathbb{C}_X} \Omega_{X,D}^\bullet \rightarrow \Omega_{X,D}^\bullet .$$

Indeed, already  $\Omega_X^\bullet(\log D) \otimes_{\mathbb{C}_X} \Omega_X^\bullet$  does not map to  $\Omega_X^\bullet$  by cup product, but to  $\Omega_X^\bullet(\log D)$ . In general, for an irreducible component  $C$  of  $D_m$  for some  $m \geq 0$  we look for a natural target for a cup product map on  $\Omega_X^\bullet(\log D) \otimes_{\mathbb{C}_X} \Omega_C^\bullet$ . This is provided by the following

**Lemma 1** *Let  $C$  be an irreducible component of  $D_m$  for some  $m$ . Then  $C$  is a smooth subvariety of  $X$ . Let  $\mathcal{I}_C \subset \mathcal{O}_X$  denote its ideal sheaf. Then  $\mathcal{I}_C \Omega_X^\bullet(\log D)$  is a subcomplex of  $\Omega_X^\bullet(\log D)$ .*

*Proof.* Let  $P \in C$ . Choose local holomorphic coordinates  $(z_1, \dots, z_n)$  on  $X$  centered at  $P$  such that  $\mathcal{I}_{C,P} = (z_1, \dots, z_k) \mathcal{O}_{X,P}$  and  $\mathcal{I}_{D,P} = (z_1 \cdots z_l) \mathcal{O}_{X,P}$  for some  $k \leq l \leq n$ . For  $\omega \in \mathcal{I}_C \Omega_X^p(\log D)_P$  write  $\omega = \sum_{i=1}^k z_i \omega_i$  with  $\omega_i \in \Omega_X^p(\log D)_P$  for  $i = 1, \dots, k$ . Then

$$d\omega = \sum_{i=1}^k z_i \left( \frac{dz_i}{z_i} \wedge \omega_i + d\omega_i \right) \in \mathcal{I}_C \Omega_X^{p+1}(\log D)_P.$$

We denote the quotient complex  $\Omega_X^\bullet(\log D) / \mathcal{I}_C \Omega_X^\bullet(\log D)$  by  $\Omega_X^\bullet(\log D) \otimes \mathcal{O}_C$ . We equip it with the filtrations  $W$  and  $F$  as a quotient of  $\Omega_X^\bullet(\log D)$ .

**Theorem 1** *The complex  $\Omega_X^\bullet(\log D) \otimes \mathcal{O}_C$  is quasi-isomorphic to  $i_C^* \mathbb{R}j_* \mathbb{C}_{X \setminus D}$  where  $i_C : C \rightarrow X$  and  $j : X \setminus D \rightarrow X$  are the inclusion maps.*

*Proof.* We have an isomorphism  $(\Omega_X^\bullet(\log D), W) \simeq (\mathbb{R}j_* \mathbb{C}_{X \setminus D}, \tau_\leq)$  in  $DF(X, \mathbb{C})$ , which by restriction to  $C$  gives an isomorphism

$$(i_C^* \Omega_X^\bullet(\log D), W) \simeq (i_C^* \mathbb{R}j_* \mathbb{C}_{X \setminus D}, \tau_\leq)$$

in  $DF(C, \mathbb{C})$ . It remains to be proven that the quotient map

$$(i_C^* \Omega_X^\bullet(\log D), W) \rightarrow (\Omega_X^\bullet(\log D) \otimes \mathcal{O}_C, W)$$

is a filtered quasi-isomorphism. To deal with this problem, note that for all  $k \geq 0$  one has the Poincaré residue map

$$R_k : Gr_k^W \Omega_X^\bullet(\log D) \rightarrow (\pi_{k-1})_* \Omega_{D_{k-1}}^\bullet[-k]$$

(where  $D_{-1} := X$ ) which is an isomorphism of complexes. It has components

$$R_I : Gr_k^W \Omega_X^\bullet(\log D) \rightarrow \Omega_{D_I}^\bullet[-k]$$

where  $I$  is a subset of  $A$  of cardinality  $k$  and  $D_I := \bigcap_{a \in I} D_a$ . It follows that

$$i_C^* Gr_k^W \Omega_X^\bullet(\log D) \simeq \bigoplus_{\#I=k} i_C^* \mathbb{C}_{D_I}[-k] \simeq \bigoplus_{\#I=k} \mathbb{C}_{D_I \cap C}[-k].$$

Claim: the image of  $\mathcal{I}_C \Omega_X^p(\log D) \cap W_k \Omega_X^p(\log D)$  under the map  $R_I$  coincides with  $\mathcal{I} \Omega_{D_I}^{p-k} + d\mathcal{I}_C \wedge \Omega_{D_I}^{p-k-1}$ .

Assuming the claim, we find that  $R_k$  induces an isomorphism

$$Gr_k^W \Omega_X^\bullet(\log D) \otimes \mathcal{O}_C \simeq \bigoplus_{\#I=k} \Omega_{D_I \cap C}^\bullet[-k].$$

Let us prove the claim. The map  $R_I$  presupposes an ordering of the set  $A$  of irreducible components of  $D$ . Write  $I = \{i_1, \dots, i_k\}$  with  $i_1 < \dots < i_k$  and choose local coordinates  $(z_1, \dots, z_n)$  on  $X$  centered at  $P \in C$  such that  $D_{i_r}$  is defined near  $P$  by  $z_r = 0$  for  $r = 1, \dots, k$  and  $\mathcal{I}_{C,P}$  is generated by  $z_j$  for  $j \in J$ . Put  $J_1 = J \cap \{1, \dots, k\}$  and  $J_2 = J \setminus J_1$ . Also suppose that  $D$  is defined near  $P$  by  $z_1 \cdots z_l = 0$ . Then  $l \geq k$  and  $J \subset \{1, \dots, l\}$ .

For  $j \in J_2$  choose  $\eta_i \in \Omega_{D_I, P}^{p-k-1}$  and  $\zeta_i \in \Omega_{D_I, P}^{p-k}$  with lifts  $\tilde{\eta}_i$  and  $\tilde{\zeta}_i$  in  $\Omega_{X, P}^{p-k-1}$  and  $\Omega_{X, P}^{p-k}$  respectively. Let

$$\omega = \sum_{j \in J_2} \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_k}{z_k} \wedge (dz_j \wedge \tilde{\eta}_i + z_j \tilde{\zeta}_j) .$$

Then  $\omega \in \mathcal{I}_C \Omega_X^p(\log D)_P \cap W_k \Omega_X^p(\log D)_P$  and

$$R_I(\omega) = \sum_{j \in J_2} (dz_j \wedge \eta_j + z_j \zeta_j) .$$

Also remark that  $R_{I'}(\omega) = 0$  if  $I \neq I' \subset A$  with  $\#I' = k$ . Hence we have the inclusion

$$\bigoplus_I (\mathcal{I} \Omega_{D_I, P}^{p-k} + d\mathcal{I}_C \wedge \Omega_{D_I, P}^{p-k-1}) \subset R_k(\mathcal{I}_C \Omega_X^p(\log D)_P \cap W_k \Omega_X^p(\log D)_P) .$$

To prove the reverse inclusion, we let  $\xi_i = \frac{dz_i}{z_i}$  if  $1 \leq i \leq l$  and  $\xi_i = dz_i$  if  $i > l$ . Also, for  $B = \{b_1, \dots, b_r\} \subset \{1, \dots, n\}$  with  $b_1 < \dots < b_r$  we put  $\xi_B = \xi_{b_1} \wedge \cdots \wedge \xi_{b_r}$ . With this notation,  $\Omega_X^p(\log D)_P$  is the free  $O_{X, P}$ -module with basis the  $\xi_B$  with  $\#B = p$ . We have

$$\mathcal{I}_C \Omega_X^p(\log D)_P = \bigoplus_{\#B=p} \mathcal{I}_{C, P} \xi_B$$

and

$$W_k \Omega_X^p(\log D)_P = \bigoplus_{\#B=p} W_k \Omega_X^p(\log D)_P \cap O_{X, P} \xi_B = \bigoplus_{\#B=p} J(B, k) \xi_B$$

where  $J(B, k)$  is an ideal of  $O_{X, P}$  generated by squarefree monomials.

For any  $B$  and any squarefree monomial  $z_E \in J(B, k)$  with  $R_I(z_E \xi_B) \neq 0$  one has  $\{1, \dots, k\} \subset B$  and  $B \cap \{k+1, \dots, l\} \subset E$ . If moreover  $z_E \in \mathcal{I}_{C, P}$  then  $J_2 \cap E \neq \emptyset$ . Choose  $j \in J_2 \cap E$ . If  $j \in B$  then

$$z_E \xi_B = \pm z_E \frac{dz_j}{z_j} \wedge \xi_{B \setminus \{j\}} = \pm dz_j \wedge z_{E \setminus \{j\}} \xi_{B \setminus \{j\}} \in d\mathcal{I}_C \wedge \Omega_X^{p-1}(\log D)_P$$

so  $R_I(z_E \xi_B) \in d\mathcal{I}_C \wedge \Omega_{D_I}^{p-k-1}$ . On the other hand, if  $j \notin B$  then

$$R_I(z_E \xi_B) = z_j R_I(z_{E \setminus \{j\}} \xi_B) \in \mathcal{I}_C \Omega_{D_I}^{p-k} .$$

**Corollary 1**  $\mathbb{H}^k(C, \Omega_X^\bullet(\log D) \otimes O_C) \simeq H^k(U_C \setminus D, \mathbb{C})$  where  $U_C$  is a tubular neighborhood of  $C$  inside  $X$ .

**Corollary 2** *One has a cohomological mixed Hodge complex  $K^\bullet(C \log D)$  on  $C$  with  $(K^\bullet(C \log D)_\mathbb{Q}, W) = (i_C^* \mathbb{R}j_* \mathbb{Q}_{X \setminus D}, \tau_{\leq})$  and  $(K^\bullet(C \log D)_\mathbb{C}, W, F) = (\Omega_X^\bullet(\log D) \otimes O_C, W, F)$ . This defines a mixed Hodge structure on  $H^k(U_C \setminus D, \mathbb{C})$ . Moreover,  $W_0 K^\bullet(C \log D) \simeq K^\bullet(C)$  so*

$$W_k H^k(U_C \setminus D) = \text{Image of } [H^k(C) \simeq H^k(U_C) \rightarrow H^k(U_C \setminus D)] .$$

The data of all  $K^\bullet(D_I \log D)$  for  $I \subset A$  give rise to a cohomological mixed Hodge complex on the semi-simplicial variety  $D_\bullet$ . We define

$$K^\bullet(D \log D) = (\pi_\bullet)_* K^\bullet(D_\bullet \log D) .$$

This is a cohomological mixed Hodge complex on  $D$  such that  $K^\bullet(D \log D)_\mathbb{Q} \simeq i_D^* \mathbb{R}j_* \mathbb{Q}_{X \setminus D}$ . It gives a mixed Hodge structure on  $H^k(U_D \setminus D)$  where  $U_D$  is a tubular neighborhood of  $D$ . The spectral sequence

$$E_1^{pq} = \mathbb{H}^q(D_p, K^\bullet(D_p \log D)) \Rightarrow \mathbb{H}^{p+q}(D, K^\bullet(D \log D))$$

can be considered as the Mayer-Vietoris spectral sequence corresponding to a covering of  $U_D \setminus D$  by deleted neighborhoods  $U_{D_i} \setminus D$ . Observe that we dispose of a natural morphism of cohomological mixed Hodge complexes  $K^\bullet(X \log D) \rightarrow K^\bullet(D \log D)$  which on cohomology induces the restriction mapping  $H^k(X \setminus D) \rightarrow H^k(U_D \setminus D)$ . We now define

$$\tilde{K}^\bullet(X, D) = \text{cone}(K^\bullet(X \log D) \rightarrow K^\bullet(D \log D))[-1] \quad (1)$$

Note that the inclusions  $K^\bullet(X) \rightarrow K^\bullet(X \log D)$  and  $K^\bullet(D) \rightarrow K^\bullet(D \log D)$  induce a morphism of cohomological mixed Hodge complexes

$$\beta : K^\bullet(X, D) \rightarrow \tilde{K}^\bullet(X, D) .$$

**Lemma 2** *The map induced by  $\beta$  on cohomology is a quasi-isomorphism.*

*Proof.* By excision, the map  $H^k(X, D) \simeq H^k(X, U_D) \rightarrow H^k(X \setminus D, U_D \setminus D)$  is an isomorphism for all  $k$ .

**Remark** As this lemma is true also locally on  $X$ , we may even conclude that  $\beta$  is a quasi-isomorphism.

**Corollary 3** *The cohomological mixed Hodge complexes  $K^\bullet(X, D)$  and  $\tilde{K}^\bullet(X, D)$  determine the same mixed Hodge structure on  $H^k(X, D)$ .*

Indeed,  $\beta$  induces a morphism of mixed Hodge structures which is an isomorphism of vector spaces, hence an isomorphism of mixed Hodge structures.

Now we proceed to the definition of the cup product on the level of complexes. Write  $\tilde{\Omega}_{X,D}^\bullet = \tilde{K}^\bullet(X, D)_\mathbb{C}$ . For each component  $C$  of  $D_\bullet$  we have a natural cup product

$$\mu_C : \Omega_X^\bullet(\log D) \otimes_\mathbb{C} \Omega_C^\bullet \rightarrow \Omega_X^\bullet(\log D) \otimes O_C .$$



These glue to give a cup product

$$\mu : \Omega_X^\bullet(\log D) \otimes_{\mathbb{C}} \Omega_{X,D}^\bullet \rightarrow \tilde{\Omega}_{X,D}^\bullet$$

which is compatible with the filtrations  $W$  and  $F$ . We conclude

**Theorem 2** *The cup product maps*

$$H^i(X \setminus D) \otimes H^j(X, D) \rightarrow H^{i+j}(X, D)$$

*are morphisms of mixed Hodge structures.*

**Remark** If  $Y$  is a complete complex algebraic variety with a closed subvariety  $Z$  such that  $Y \setminus Z$  is smooth, then there exists a proper modification  $f : X \rightarrow Y$  such that  $X$  is smooth,  $f$  maps  $X \setminus f^{-1}(Z)$  isomorphically to  $Y \setminus Z$  and  $D := f^{-1}(Z)$  is a divisor with strict normal crossings on  $X$ . Then one has isomorphisms of mixed Hodge structures  $H^i(Y \setminus Z) \rightarrow H^i(X \setminus D)$  and  $H^j(Y, Z) \rightarrow H^j(X, D)$  so that case is reduced to the strict normal crossing case.

**Remark** The restrictions of  $K^\bullet(X \log D)$ ,  $K^\bullet(X, D)$  and  $\tilde{K}^\bullet(X, D)$  are all equal to  $K^\bullet(X \setminus D)$ . Now consider the following situation:  $Y$  is a complete complex algebraic variety with closed subvarieties  $Z$  and  $W$  such that  $Y \setminus (Z \cup W)$  is smooth and  $Z \cap W = \emptyset$ . Then there is a cup product

$$H^i(Y \setminus Z, W) \otimes H^j(Y \setminus W, Z) \rightarrow H^{i+j}(Y, Z \cup W)$$

which is a morphism of mixed Hodge structures and induces a perfect duality if  $i + j = 2 \dim(Y)$ . The proof uses a proper modification  $f : X \rightarrow Y$  such that  $X$  is smooth,  $f$  maps  $X \setminus f^{-1}(Z \cup W)$  isomorphically to  $Y \setminus (Z \cup W)$  and  $D = f^{-1}(Z)$  and  $E = f^{-1}(W)$  are divisors with normal crossings on  $X$ . By a glueing process one obtains cohomological mixed Hodge complexes  $K^\bullet(X \log D, E)$  etc. and a cup product map

$$K^\bullet(X \log D, E)_{\mathbb{C}} \otimes K^\bullet(X \log E, D)_{\mathbb{C}} \rightarrow \tilde{K}^\bullet(X, D \cup E)_{\mathbb{C}}$$

which is compatible with  $W$  and  $F$ .

This answers a question raised to me by V. Srinivas.

## 4 The extraordinary cup product

Again, let  $X$  be a complete smooth complex algebraic variety and let  $D$  be a divisor with strict normal crossings on  $X$ . The local cohomology groups  $H_D^k(X) = H^k(X, X \setminus D)$  get a mixed Hodge structure using the cohomological mixed Hodge complex

$$K_D^\bullet(X) = \text{cone}[K^\bullet(X) \xrightarrow{u} K^\bullet(X \log D)] \quad (2)$$

but by excision we may as well take

$$\tilde{K}_D^\bullet(X) = \text{cone}[K^\bullet(D) \xrightarrow{v} K^\bullet(D \log D)] \quad (3)$$

Observe that the morphisms  $u_{\mathbb{C}}$  and  $v_{\mathbb{C}}$  are injective, even after taking  $Gr_F Gr^W$ , so we have bifiltered quasi-isomorphisms

$$(K_D^\bullet(X)_{\mathbb{C}}, W, F) \rightarrow (\text{coker}(u_{\mathbb{C}}), W, F)[-1]$$

and

$$\tilde{K}_D^\bullet(X)_{\mathbb{C}}, W, F) \rightarrow (\text{coker}(v_{\mathbb{C}}), W, F)[-1] .$$

Moreover, the natural cupproduct

$$K^\bullet(X \log D)_{\mathbb{C}} \otimes_{\mathbb{C}} K^\bullet(D)_{\mathbb{C}} \rightarrow K^\bullet(D \log D)_{\mathbb{C}}$$

maps  $K^\bullet(X)_{\mathbb{C}} \otimes_{\mathbb{C}} K^\bullet(D)_{\mathbb{C}}$  to  $K^\bullet(D)_{\mathbb{C}}$ , so induces a cup product map

$$\text{coker}(u_{\mathbb{C}}) \otimes_{\mathbb{C}} K^\bullet(D)_{\mathbb{C}} \rightarrow \text{coker}(v_{\mathbb{C}}) \quad (4)$$

which is compatible with the filtrations  $W$  and  $F$ . Hence we conclude

**Theorem 3** *The extraordinary cup product map*

$$H_D^i(X) \otimes H^j(D) \rightarrow H_D^{i+j}(X)$$

*is a morphism of mixed Hodge structures.*

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